Differential Geometry and its Applications 28 (2010) 355–358

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

# A characterization of submanifolds of extrinsic symmetric spaces

J. Eschenburg *, R. Tribuzy

*Institut für Mathematik, Universität Augsburg, D-86135 Augsburg, Germany*
*Departamento de Matemática, Universidade Federal do Amazonas, Av. Gen Rodrigo Otávio, 3000, 69077000 Manaus, Brazil*

ARTICLE INFO

*Article history:*
Received 20 September 2009
Communicated by L. Vanhecke

*MSC:*
primary 53B25, 53C35
secondary 53C05

*Keywords:*
Vector subbundles
Connections
Gauss equations

ABSTRACT

A submanifold $M \subset \mathbb{R}^n$ lies in the sphere $\mathbb{S}^{n-1}$ iff it carries a parallel umbilic normal vector field. We extend this theorem by replacing the sphere $\mathbb{S}^{n-1}$ by an arbitrary extrinsic symmetric space $S \subset \mathbb{R}^n$.



## 1. Introduction

The present article is motivated by two classical problems in submanifold geometry, reduction of codimension and umbilicity. By Erbacher [1], the codimension of an immersion $f : M \to \mathbb{R}^n$ can be reduced iff there is a parallel bundle $E \subset M \times \mathbb{R}^n$ containing $TM$. On the other hand, an immersion $f$ takes values in the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ iff there is a normal vector field $\xi$ with

$$\nabla^\perp \xi = 0, \qquad \langle \alpha(v, w), \xi \rangle = \langle v, w \rangle \tag{1}$$

where $\nabla^\perp$ is the connection in the normal bundle and $\alpha$ is the second fundamental form of $f$, cf. [5]. What happens if we replace the sphere by any extrinsic symmetric space $S \subset \mathbb{R}^n$? These are natural generalizations of the sphere in many respects. Clearly we need a bundle $E \subset M \times \mathbb{R}^n$ which contains $TM$ and carries the structure of $TS$, and some condition on the $E^\perp$-part of $\alpha$ replacing (1) will be needed.

**Notation.** We do not distinguish between a vector bundle $E$ and its space of sections $\Gamma E$. By $a \in E$ we mean that $a$ is a (locally defined) section of $E$, and $v \in TM$ says that $v$ is a (local) tangent vector field of $M$.

## 2. The result

Let $S \subset \mathbb{R}^n$ be a full extrinsically symmetric space [4], i.e. a compact submanifold whose second fundamental form $\beta^S : TS \otimes TS \to NS$ is parallel and onto. Consider a submanifold $M$ of $S$ and put $E = TS|_M$. The restriction of $\beta^S$ is a parallel bundle map $\beta : E \otimes E \to E^\perp$ which is related to the "second fundamental form" $\beta^E(v, a)$ of the bundle $E$ as follows:

* Corresponding author.
*E-mail addresses:* eschenburg@math.uni-augsburg.de (J. Eschenburg), tribuzy@pq.cnpq.br (R. Tribuzy).

0926-2245/$ – see front matter 
doi:10.1016/j.difgeo.2010.01.002

$$\beta^E(v,a) := (\partial_v a)^{E^\perp} = \beta(v,a) \tag{2}$$

for all $v \in TM$ and $a \in E$. It is our aim to show that these properties are already sufficient for a submanifold $M$ of $\mathbb{R}^n$ to lie actually in $S$:

**Theorem 1.** *Let $M$ be a submanifold of some Euclidean space $\mathbb{R}^n$ which also contains a full extrinsic symmetric space $S$. Suppose that there is a subbundle $E \subset M \times \mathbb{R}^n$ with $TM \subset E$ and a parallel bundle homomorphism $\beta : E \otimes E \to E^\perp$ satisfying (2) such that $\beta$ is congruent to the second fundamental form $\beta^S$ at some point. Then $M$ is contained in $S$, up to a rigid motion of $\mathbb{R}^n$.*

**Remark 1.** In the case where $S$ is the sphere, $S = \mathbb{S}^{n-1}$, the assumptions of our theorem are those of the umbilicity theorem (1). In fact, in this case $E = \xi^\perp$ for some normal vector field $\xi$ on $M$ and $\beta(a,b) = \langle a,b\rangle\xi$ is always parallel. Condition (2) says $-\langle\partial_v\xi, a\rangle = \langle\partial_v a, \xi\rangle = \langle v, a\rangle$ for all $v \in TM$, $a \in E$. In particular $\langle\partial_v\xi, a\rangle = 0$ for every $a \perp TM$ which shows that $\xi$ is a parallel normal vector field with $\langle\alpha(v,w),\xi\rangle = -\langle\partial_v\xi, w\rangle = \langle v, w\rangle$ for all $v, w \in TM$. Vice versa, if $M \subset \mathbb{R}^n$ carries a parallel normal vector field $\xi$ with $-\langle\partial_v\xi, w\rangle = \langle\alpha(v,w),\xi\rangle = \langle v,w\rangle$, we have $\partial_v\xi = (\partial_v\xi)^{TM} = -v$ and hence $(\partial_v a)^{E^\perp} = \langle\partial_v a,\xi\rangle\xi = -\langle\partial_v\xi, a\rangle\xi\rangle = \langle v,a\rangle\xi = \beta(v,a)$, as in the assumptions of our theorem.

**Remark 2.** In [2] we determined under some genericity assumptions, which maps $\nu$ from a Riemannian manifold $M^m$ to the Grassmannian $G_m(\mathbb{R}^n)$ can be the Gauss maps of an isometric immersion $M \hookrightarrow \mathbb{R}^n$. A related question is if we can see from $\nu$ when $M$ actually lies in a sphere or in an extrinsic symmetric space within $\mathbb{R}^n$. Our theorem gives an answer: it is necessary and sufficient that the bundle $\tau = \nu^\perp$ can be extended to a bundle $E \supset \tau$ with the properties as in the assumptions of our theorem.

## 3. The proof

The main idea of the proof is to apply the general existence and uniqueness theorem for submanifolds of symmetric spaces, cf. [3]: If $S$ is a symmetric space and $f : M \to S$ an immersion, then the differential of $f$ is a bundle homomorphism $F = df : TM \to E := f^*TS$ with

$$(\nabla_v F)w = (\nabla_w F)v, \tag{3}$$

$$R^E(v,w)a = R^S(Fv, Fw)a \tag{4}$$

for all $v, w \in TM$ and $a \in E$, where $R^S$ is the curvature tensor of $S$ and $R^E$ the curvature tensor of the vector bundle $E$. Vice versa, if data $(E, \nabla, R^S, F)$ are given, where $E$ is a vector bundle over $M$ with a metric connection $\nabla$ and $R^S : E \otimes E \otimes E \to E$ a $\nabla$-parallel isomorphic copy of the curvature tensor of $S$ and $F : TM \to E$ an injective vector bundle homomorphism with (3), (4), then $F$ is the differential of an immersion $\tilde f : M \to S$ or more precisely, there exist an immersion $\tilde f : M \to S$ and a parallel bundle isometry $\Phi : \tilde f^*TS \to E$ carrying $R^S$ on $\tilde f^*TS$ into $R^S$ on $E$ such that

$$d\tilde f = \Phi \circ F. \tag{5}$$

In our case, $S$ is an extrinsic symmetric space in $\mathbb{R}^n$, and the parallel curvature tensor $R^S$ enjoys a parallel "square root" taking care of the normal geometry as well: the second fundamental form $\beta : E \otimes E \to E^\perp$ which is related to $R^S$ by the (quadratic) Gauss equation

$$\bigl\langle R^S(c,d)a, b\bigr\rangle = \bigl\langle\beta(c,b), \beta(d,a)\bigr\rangle - \bigl\langle\beta(c,a), \beta(d,b)\bigr\rangle. \tag{6}$$

Now suppose that we have got an immersion $f : M \to \mathbb{R}^n$ with the properties assumed in the theorem. We apply the existence and uniqueness theorem to $F = df : M \to E$. Eq. (3) is obvious from $F = df$. In order to prove (4), one needs a version of the Gauss equation for subbundles:

**Lemma 2.** *Let $M$ be a manifold and $E \subset M \times \mathbb{R}^n$ a vector bundle, equipped with the projection connection $\nabla_v a = (\partial_v a)^E$ where $a \in E$ and $v \in TM$. Let $\beta^E : TM \otimes E \to E^\perp$ be its "second fundamental form"*

$$\beta^E(v,a) = (\partial_v a)^{E^\perp} \tag{7}$$

*with $v \in TM$ and $a \in E$. This is related to the curvature tensor $R^E$ of $(E,\nabla)$ as follows:*

$$\bigl\langle R^E(v,w)a, b\bigr\rangle = \bigl\langle\beta^E(v,b), \beta^E(w,a)\bigr\rangle - \bigl\langle\beta^E(v,a), \beta^E(w,b)\bigr\rangle. \tag{8}$$

**Proof.** The lemma is proved like Gauss equations for submanifolds: From the decomposition $\partial_w a = \nabla_w a + \beta^E(w,a)$ we obtain

$$\langle\partial_v\partial_w a, b\rangle = \langle\nabla_v\nabla_w a, b\rangle - \bigl\langle\beta^E(w,a), \beta^E(v,b)\bigr\rangle$$

and hence

$$0 = \langle R^\partial(v,w)a,b\rangle = \langle R^E(v,w)a,b\rangle - \langle \beta^E(w,a),\beta^E(v,b)\rangle + \langle \beta^E(v,a),\beta^E(w,b)\rangle$$

which finishes the proof. □

Now putting $c = v$ and $d = w$ in the Gauss equation (6) and using $\beta = \beta^E$ for these entries (cf. (2)), we obtain (4) by comparing (6) and (8). Using the general existence theorem (5) we obtain an immersion $\tilde f : M \to S \subset \mathbb{R}^n$ with $df = \Phi \circ d\tilde f$ for some parallel and isometric bundle isomorphism $\Phi : \tilde f^* TS \to E$; in particular, the two immersions $f$ and $\tilde f$ are (intrinsically) isometric. It remains to show that they just differ by an isometry of the ambient space $\mathbb{R}^n$.

In order to prove the congruence of $\tilde f$ and $f$ we extend $\Phi$ to an isometric bundle isomorphism $\hat\Phi$ of

$$V = M \times \mathbb{R}^n = E + E^\perp = \tilde E + \tilde E^\perp$$

where $\tilde E := \tilde f^* TS$. This is possible by the geometry of extrinsic symmetric spaces:

**Lemma 3.** *There is an extension of $\Phi : \tilde E \to E$ to a bundle isometry $\hat\Phi : V \to V$ such that*

$$\hat\Phi\big(\tilde\beta(a,b)\big) = \beta(\Phi a, \Phi b) \tag{9}$$

*for all $a, b \in \tilde E$, where $\tilde\beta : \tilde E \otimes \tilde E \to \tilde E^\perp$ is the second fundamental form of $S$ along $\tilde f$.*

**Proof.** Since every normal vector $\xi \in \tilde E^\perp$ is of the form $\tilde\beta(a,b)$ for some $a, b \in \tilde E$, it suffices to show that $\hat\Phi$ is well defined by (9), i.e. for all $a, b, a', b' \in \tilde E$

$$\tilde\beta(a,b) = \tilde\beta\big(a',b'\big) \quad\Rightarrow\quad \beta(\Phi a,\Phi b) = \beta\big(\Phi a',\Phi b'\big). \tag{10}$$

By assumption of our theorem, $\beta$ and $\tilde\beta$ are congruent at some point $p \in M$. Hence we may assume $E_p = \tilde E_p$ and $\Phi_p = I$, and (10) is obvious at $p$. At any other point $q \in M$ we consider a curve $\gamma$ joining $p$ to $q$ and the parallel displacements of the various bundles along this curve: $\tau, \tilde\tau$ along $E, \tilde E$ and $\tau^\perp, \tilde\tau^\perp$ along $E^\perp, \tilde E^\perp$. Since $\Phi, \tilde\beta, \beta$ are parallel, its values at $p$ and $q$ are joined by parallel displacements: We have $\Phi_q \tilde\tau = \tau\Phi_p$ and $\tau^*\beta = \tau^\perp\beta$, $\tilde\tau^*\tilde\beta = \tilde\tau^\perp\tilde\beta$, hence

$$\tilde\beta_q(\tilde\tau a, \tilde\tau b) = \tilde\tau^\perp\tilde\beta_p(a,b), \qquad \beta_q(\Phi_q\tilde\tau a, \Phi_q\tilde\tau b) = \tau^\perp\beta_p(\Phi_p a, \Phi_p b)$$

for all $a, b \in \tilde E_p$. This proves (10) at $q$. □

**Lemma 4.** *$\hat\Phi : V \to V$ is parallel with respect to $\partial$.*

**Proof.** There are several connections on $V$, the trivial connection $\partial$ and the two projection connections $\nabla, \tilde\nabla$ for the decompositions $V = E + E^\perp$ and $V = \tilde E + \tilde E^\perp$; e.g. we have $\nabla(a+\xi) = (\partial a)^E + (\partial\xi)^{E^\perp}$ for $a \in E$ and $\xi \in E^\perp$. We know already that $\hat\Phi$ is parallel with respect to the connections $\tilde\nabla$ on the domain and $\nabla$ on the range. For the $E$-component this is due to the parallelity of $\Phi$, see (5), and for the $E^\perp$-part it follows from (9) and the parallelity of $\tilde\beta$, $\beta$ and $\Phi$.

To prove $\partial$-parallelity of $\hat\Phi$ we have to see that $\hat\Phi$ carries $\tilde A = \partial - \tilde\nabla$ onto $A = \partial - \nabla$. From $\partial_v a = \nabla_v a + \beta(v,a)$ and $\partial_v\xi = \nabla_v\xi - A_\xi v$ we find

$$A(v,a) = \beta(v,a), \qquad A(v,\xi) = -A_\xi v \tag{11}$$

where $A_\xi = -(\partial\xi)^E$ is the "Weingarten map" of $E$, and similar for $\tilde A$. Using (9) we have

$$\hat\Phi\tilde A(v,a) = \hat\Phi\tilde\beta(v,a) = \beta(v,\Phi a) = A(v,\Phi a) \tag{12}$$

where we consider $TM$ both as a subbundle of $E$ (using $df$) and of $\tilde E$ (using $d\tilde f$). Moreover, for $a \in \tilde E$ and $v \in TM$ and $\xi \in \tilde E^\perp$ we have

$$\big\langle A(v,\hat\Phi\xi),\Phi a\big\rangle = \big\langle \beta(v,\Phi a),\hat\Phi\xi\big\rangle = \big\langle \hat\Phi\beta(v,a),\hat\Phi\xi\big\rangle = \big\langle \beta(v,a),\xi\big\rangle = \big\langle \tilde A(v,\xi),a\big\rangle$$

from which we conclude $\Phi^{-1}A(v,\hat\Phi\xi) = \tilde A(v,\xi)$ and hence

$$\Phi\tilde A(v,\xi) = A(v,\hat\Phi\xi). \tag{13}$$

Thus $\hat\Phi\tilde A(v,x) = A(v,\hat\Phi x)$ for any $x \in V$, and therefore

$$\partial_v\hat\Phi x = \nabla_v\hat\Phi x + A(v,\hat\Phi x) = \hat\Phi\big(\tilde\nabla_v x + \tilde A(v,x)\big) = \hat\Phi\partial_v x. \qquad \square$$

Now $\hat\Phi : V \to V$ is a constant orthogonal map relating $df$ and $d\tilde f$, see (5), and the proof of the theorem is finished.

## References

[1] J.A. Erbacher, Reduction of codimension of an isometric immersion, J. Differential Geom. 5 (1971) 333–340.
[2] J.-H. Eschenburg, B. Kruglikov, V. Matveev, R. Tribuzy, Compatibility of Gauß maps with metrics, Differential Geom. Appl. 28 (2) (2010) 228–235, arXiv:0903.3225.
[3] J.-H. Eschenburg, R. Tribuzy, Existence and uniqueness of maps into affine homogeneous spaces, Rend. Semin. Mat. Univ. Padova 89 (1993) 11–18.
[4] D. Ferus, Symmetric submanifolds of Euclidean space, Math. Ann. 247 (1980) 81–93.
[5] S.T. Yau, Submanifolds with constant mean curvature, I, Amer. J. Math. 96 (1974) 346–366.